Modeling Earthquake Effects on Buildings

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Modeling Building Vibrations

- Attempt to model the effect of an earthquake on a multistory building

- Described through *n* second-order differential equations

- How?
  - The differential equations for the model are obtained by **competition**: the Newton's second law force is set equal to the sum of the Hooke's forces and the external force due to the earthquake wave.

Specific Constraints

- Each floor is considered a point mass (center-of-mass).
  - Masses: $m_1, \dots, m_n$
- Each floor is restored to its equilibrium position by a linear restoring force or Hooke's force −k(elongation).
  - The Hooke's constants: $k_1, \dots, k_n$
- The locations of masses representing the 'n' floors are $x_1, \dots, x_n$.
  - The equilibrium position is $x_1 = \cdots = x_n = 0$
- Dampening effects are ignored.

Vibrations and Energy Exchange

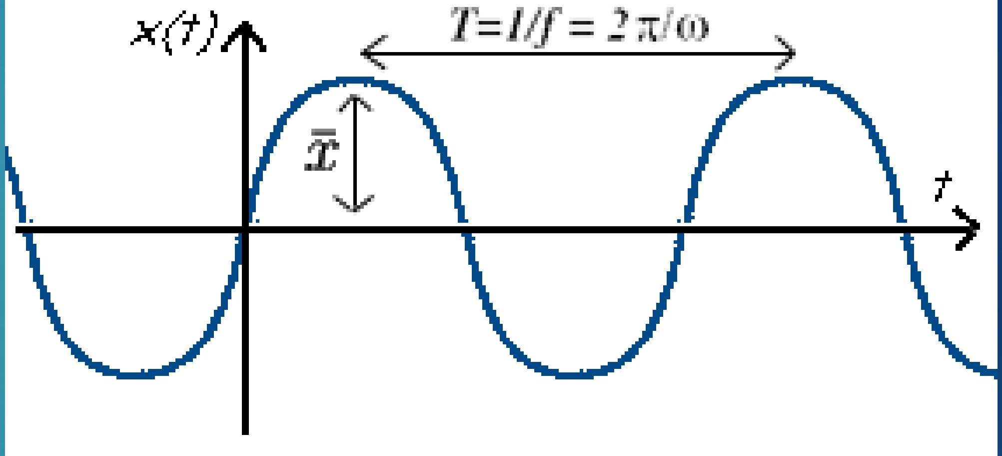
- The system works similar to a Spring-Mass System.
- The equations for a floor depend only upon the neighboring floors.
  - Exceptions: Top and Bottom
- Important Note, the model mathematically demonstrates resonance frequencies ONLY

What is Resonance?

- The model discusses resonance only: Mechanical Resonance (MR)
  - Defined as: the tendency of a mechanical system to respond at greater amplitudes when the frequency (omega) of its oscillations matches the system's natural frequency of vibration (resonance frequency).

Needed Functions and Equations

- Horizontal earthquake oscillation
  - $F(t) = F_0 \cos(\omega t)$
- Natural frequency of oscillation
  - $\sqrt{\frac{k}{m}}$
- External force: $H(t)$
  - $E(t) = -mF''(t)$
  - $H(t) = -\omega^2 F_0 \cos(\omega t) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$



Matrix Interpretations

- M : $(n \times n)$ mass matrix
  - Positive definite as every floor has a positive mass
- K : $(n \times n)$ stiffness matrix
  - Positive definite as building is not free floating
- X : Vector of the displacement for each story
- $Mx''(t) + Kx(t) = H(t)$
  - $MX'' = -KX + H$

Matrix Interpretations cont'd

- $X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}; \quad M = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{pmatrix}; \quad H = \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{pmatrix}$

- $K = \begin{pmatrix} -(k_1 + k_2) & k_2 & 0 & 0 & \cdots & 0 & 0 \\ k_2 & -(k_2 + k_3) & k_3 & & \cdots & 0 & 0 \\ 0 & k_3 & -(k_3 + k_4) & k_4 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & k_n & -k_n \end{pmatrix}$

- This system is a non-autonomus system.

Two-Story Example

- Each floor has mass = 5000 kg.
- Restoring force constant: k=10,000 kg/s^2
- System:
  - $x_1'' = -4x_1 + 2x_2$
  - $x_2'' = 2x_1 - 2x_2$

- Wave of the hand:
  - $x_1(t) = c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) + c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)$
  - $x_2(t) = \frac{1}{2}(4 - \omega_1^2)c_1 \cos(\omega_1 t) + \frac{1}{2}(4 - \omega_1^2)c_2 \sin(\omega_1 t) + \frac{1}{2}(4 - \omega_2^2)c_3 \cos(\omega_2 t) + \frac{1}{2}(4 - \omega_2^2)c_4 \sin(\omega_2 t)$

Two-Story Cont'd

- $X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$; $M = \begin{pmatrix} 5000 & 0 \\ 0 & 5000 \end{pmatrix}$;
- $K = \begin{pmatrix} -20000 & 10000 \\ 10000 & -10000 \end{pmatrix}$
- $X'' = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix} X$
- Eigenvalues: $\begin{bmatrix} \sqrt{5} - 3 \\ -3 - \sqrt{5} \end{bmatrix} = \begin{bmatrix} -0.763932 \\ -5.23607 \end{bmatrix}$; both negative

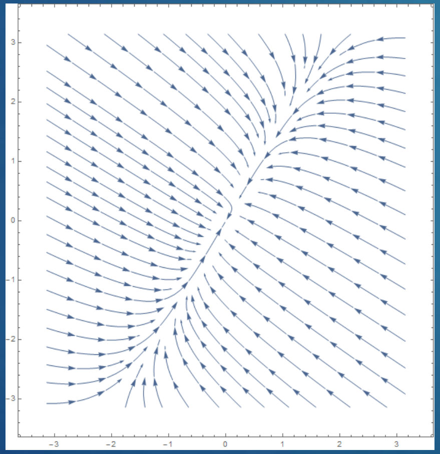
Example cont'd

λ_n	-0.763932	-5.23607
ω_n	0.874032	2.28825
T_n	7.18874	2.74585

- From these value we can conclude that as long as the frequency is not close to the eigenvalues the building does not seem to be in any danger of developing resonance during a typical earthquake whose period is within T.

Autonomous Case

- By solving Det(m^{-1} K ω^2 I) = 0 for ω reveals resonance of building by floor
  - Derived from the 2nd order equation in autonomous form
- The autonomous case also reveals a "snapshot" of the floor movement at some time, t.



- As one might expect, since the eigenvalues are negative, the linear system is asymptotically stable, which physically makes since – The system *is* always attempting to return to its equilibrium position, 0. The only critical point in the system.

What the CENSORED does all this mean?

- System is asymptotically stable as long as the frequency of the forcing term **F** is not close to one of the natural frequencies of the building. (The displacement of the floors do not blow up mathematically)
- Specifically an earthquake of proper frequency, and sufficient duration, can demolish a single floor and bringing the building down.
- Using the Richter Scale formulae, the resonance frequencies of the building can be used to graph the magnitude and distance estimates an earthquake may need to destroy a building.